

Application of Bayes' Theorem to Statistical Analysis of Small Sample Data

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1. Foreword

In recent years, a probabilistic approach is increasingly needed as mechanical products are expected to provide higher functionality. This is not unrelated to the fact that, as mechanical products provide higher functionality, the consequences of any accident tend to be more serious. A typical example would be an on-board composite hydrogen tank stored under high pressure. In his examination of standards development, for example, G. W. Mair proposes to introduce a probabilistic approach to the consideration of safety margins⁽¹⁾. The present author introduces the concept of a probabilistic approach, which is often discussed in terms of GTR and other standards, as an approach to safety standards for composite hydrogen tanks for fuel cell vehicles⁽²⁾. The current paper provides an overview from the perspective of reliability design that the author has experienced outside the automotive field. This paper has mentioned the historical background, details of the method, advantages and disadvantages, examples of use in other fields, and the possibility of use in fuel cell vehicles. Following this, an important action would be to start an evidence-based discussion of safety margins for bursting strength, with specific application of a probabilistic approach to the current situation in Japan. An essential part of this process is the organization of statistical data. The basic statistics involved include the mean value and variance of the bursting pressure of hydrogen tanks. However, obtaining basic statistics of hydrogen tank bursting strength experimentally is not easy in terms of cost and labor. Moreover, considering the severity of the results, the need for a higher degree of safety performance means that the probability of bursting is extremely small, which also makes it a challenge to take a large sample size. If a very large amount of data could be obtained, it might not be such a challenge to evaluate the basic statistics of the population with a high degree of accuracy based on so-called sample statistics. However, in the case of composite hydrogen tanks, if large sample sizes are not expected, then the application of sample statistics is generally a challenge. In such cases, there is an approach called Bayesian statistics that is effective for evaluating small-sample statistics. For example, Bayes' theorem is used for the statistical processing of maintenance data in risk-based inspections of plant pressure equipment^{(3), (4)}. This is related to the fact that one cannot obtain large samples of maintenance data. This paper

summarizes the basic ideas of Bayes' theorem as well as examples, applications, and precautions related to applying the theorem.

2. Characteristics of Bayesian Statistics

Before explaining Bayes' theorem in detail, this chapter discusses its characteristics. When an evaluation of the basic statistics involved in the breakage of a mechanical product is required, it is advisable to first consider applying Bayes' theorem in the following situations.

1. The event in question is very infrequent
2. Causing the event by experimentation would require excessive cost and labor
3. The event may be transitory

In such situations, one cannot easily increase the sample size rapidly and it would be challenging to apply sample statistics that assume a large sample size. Additional information follows concerning the third situation above. For example, when considering plant maintenance, the damage status of piping and pressure vessels depends also on operating conditions such as internal pressure. If operating conditions are changed at some point in time, the basic statistics related to the probability of damage may also subsequently change. In addition, design specifications are likely to be modified during the development phase of products like hydrogen tanks. In this case as well, while the basic statistics on damage probability may be predicted to change, the sample size will never increase if sampling restarts each time there is a change in design specifications. Even in challenging situations like these, Bayesian statistics has the following characteristics that allow it to handle such situations to some extent.

1. For statistical processing, it uses not only the information that there was damage, but also information that there was no damage.
2. In identifying the statistical distribution, this method updates the certainty of the parameter from a small sample size.
3. The certainty of the parameter is determined at the starting point by such factors as experienced values, literature, and subjectivity.

In sample statistics, a parameter such as the mean is

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evaluated as a single value by statistical processing from sample data. In Bayesian statistics, however, the parameter is not evaluated as a single value, but is rather treated as something ambiguous; that is, a level of certainty. In extreme cases, certainty may be determined subjectively as a starting point, which is why it is called subjective probability. Although it may seem unwise to allow subjectivity to enter into the statistical process, in Bayesian statistics, the statistics are updated each time sample data is obtained, so that certainty gets dependably closer to the correct answer, and is thus expected to eventually converge to the true value. Even if the event is transitory as described above, Bayesian statistics can track the transition of the event, since processing is accomplished by updating. It is wise, however, to have a good understanding of the principles of Bayesian statistics, because if not applied appropriately, the efficiency of such convergence will not occur.

3. Basic Principles of Bayesian Statistics

3.1. Approach

If the object of interest is a pressure tank, assume that its bursting pressure is normally distributed. The probability density function with a normal distribution is expressed as:

$$f(x|\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \quad (1)$$

where x is the random variable for bursting pressure, and there are two parameters: mean μ and variance σ^2 . $f(x|\mu, \sigma)$ is a probability density function with the parameters μ and σ . A normal distribution with mean μ and variance σ^2 is sometimes expressed as $N(\mu, \sigma^2)$. To identify the basic statistics of bursting pressure is to find μ and σ^2 . If a sufficiently large sample size n can be obtained, the sample mean $\hat{\mu}$ and unbiased sample variance $\hat{\sigma}^2$ can be evaluated from the following equations.

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i \quad (2)$$

$$\hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \hat{\mu})^2 \quad (3)$$

If a larger sample size can be obtained, these parameters can be evaluated with greater accuracy. When this is not expected, however, one should consider using Bayesian statistics. Here, for simplicity, the variance σ^2 is assumed to be known, and only the mean value μ is unknown. The way to estimate this is explained as follows. The approach can be applied even if both parameters are unknown, but that process will be discussed later.

The evaluation using Eq. (2) establishes one sample mean value, and there is no room for subjectivity in the procedure. For that reason, it can be called an objective procedure. Bayesian statistics, in contrast, adds uncertainty to the estimated quantity μ . In other words, it does not

assign a single definitive value. If new experimental data is obtained, the result indicating whether a pressure vessel did or did not burst under certain conditions is an objective fact. Bayesian statistics involves updating the uncertainty of an estimated quantity based on a few objective facts. This is called Bayesian updating. While one can say that Bayesian updating does not suddenly yield the correct answer for the parameter, it does approach the correct answer. Uncertainty before updating is called the prior distribution, and uncertainty after updating is called the posterior distribution. Figure 1 illustrates the approach conceptually. The vertical axis is the probability density.

For example, when the left side of Fig. 1 is the prior distribution of uncertainty and μ_0 is the correct answer for the parameter, the figure shows how, based on the results of a single burst test, the posterior distribution of uncertainty gathers near μ_0 based on the Bayesian update results. The greater the number of burst test samples, the better the convergence results that can be expected, and increasing the number of times tests are repeated will yield results asymptotically approaching the correct answer. The first prior distribution to be set is called the initial prior distribution. If the initial prior distribution is off the mark, there is unlikely to be good convergence performance, so it is important to set it carefully. The initial prior distribution can be set based on the results of similar products in the past when available, on the opinions of experienced persons, or on information found in the literature. This initial prior distribution is sometimes called subjective probability, because in any case it is not given definitively but is determined subjectively in a sense. If starting with no information beforehand, the uncertainty is distributed over a wide area, and is called an uninformative prior distribution. In Fig. 1, the prior distribution is set as a continuous distribution, but it can also be set as discrete values, as in Fig. 2. That is, instead of expressing the

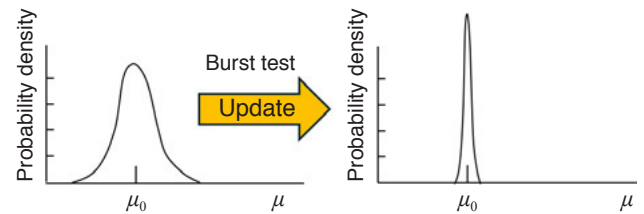


Fig. 1 Principle of Bayesian update for continuous distribution

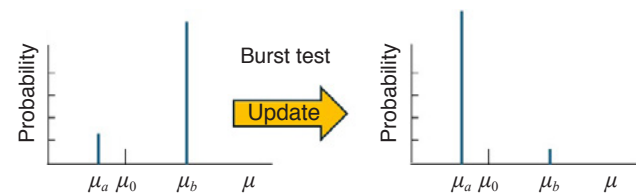


Fig. 2 Principle of Bayesian update for discrete values

uncertainty of the mean as a continuous distribution, it is estimated as a discrete value in the form of either μ_a or μ_b .

In this figure, in the prior distribution shown on the left, uncertainty confidence values for the μ_a and μ_b values estimated are set discretely. In this example, μ_a has a lower confidence level than μ_b , even though μ_a is closer to μ_0 , the true value. After sample values are obtained, the Bayesian update results in a higher confidence level for μ_a , and the true value can be determined to be closer to μ_a .

3.2. Basic Principles

3.2.1. Estimating discrete values

Suppose that a parameter one wishes to estimate by Bayes' theorem, such as a population parameter, is θ . When the parameter is θ , the probability density function for an observed event x , such as bursting, is expressed as $f(x|\theta)$. In ordinary sample statistics, the sample value $\hat{\theta}$ is evaluated from sample data by statistical procedures, leaving no room for subjectivity. Bayesian statistics, on the other hand, sets a value for θ subjectively and also assigns a confidence level for the uncertainty of that value. Here, for simplicity, two discrete values (θ_1 and θ_2) have been set. P_1 and P_2 represent the confidence level for each value at the start. Bayes' theorem assumes that observed events necessarily arise based on the value of θ_1 or θ_2 . This is to say, the condition $P_1 + P_2 = 1$ needs to be satisfied. For example, if there is a high degree of confidence that the parameter is θ_1 , one may set $P_1 = 0.8$ and $P_2 = 0.2$. Observed events, such as bursting, are now written as A . At this time, the logical relationship can be represented by a Venn diagram as in Fig. 3. Here, C_1 indicates that the observed event is caused by θ_1 and C_2 indicates that it is caused by θ_2 . From now on, the probability of an event will be expressed as P (event). The probability that the causal event is C_1 and that the event A is observed can be expressed by the following equation using the basic properties of a Venn diagram.

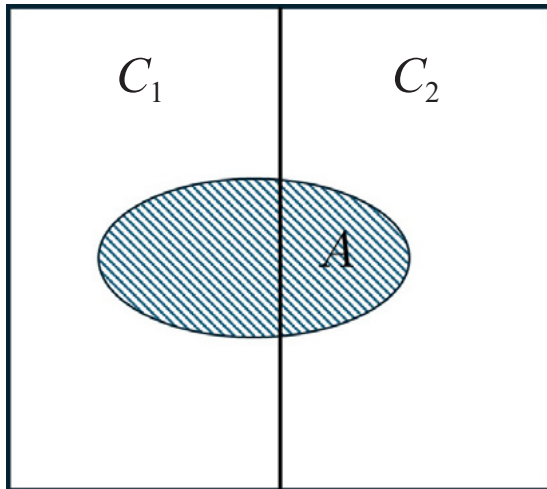


Fig. 3 Venn diagram

$$\begin{aligned} P(C_1 \wedge A) &= P(C_1)P(A|C_1) \\ &= P(A)P(C_1|A) \end{aligned} \quad (4)$$

where $\wedge$ denotes the AND logical relationship and $P(A|C_1)$ denotes the conditional probability that A is observed under the condition of cause C_1 . By transforming Eq. (4), one can obtain:

$$P(C_1|A) = \frac{P(C_1)P(A|C_1)}{P(A)} \quad (5)$$

$P(A)$ can be written as:

$$P(A) = P(C_1)P(A|C_1) + P(C_2)P(A|C_2) \quad (6)$$

Therefore, substituting this into Eq. (5) can be expressed as:

$$P(C_1|A) = \frac{P(C_1)P(A|C_1)}{P(C_1)P(A|C_1) + P(C_2)P(A|C_2)} \quad (7)$$

The components on the right side of Eq. (7) are given as below in the above case.

$$\begin{aligned} P(C_1), P(C_2) &\Leftrightarrow P_1, P_2 \\ P(A|C_1), P(A|C_2) &\Leftrightarrow f(x|\theta_1), f(x|\theta_2) \end{aligned} \quad (8)$$

Thus, all quantities on the right side are given quantities. On the other hand, $P(C_1|A)$ on the left side indicates the probability of C_1 under the constraint that there was an observed event A . It therefore indicates the degree of confidence in C_1 after incorporating the observed event. Thus, $P(C_1|A)$ is the posterior probability after the Bayesian update. $P(C_2|A)$ is found similarly:

$$P(C_2|A) = \frac{P(C_2)P(A|C_2)}{P(C_1)P(A|C_1) + P(C_2)P(A|C_2)} \quad (9)$$

$P(C_1|A)$ and $P(C_2|A)$ indicate that the degree of confidence in C_1 and C_2 has been updated from $P(C_1)$ and $P(C_2)$ based on the observed event A . In this manner, Bayesian statistics updates the degree of confidence based on observed events. When an observed event is subsequently obtained, this $P(C_1|A)$ and $P(C_2|A)$ can be used as prior probabilities for the next time, and the same procedure can then be used to continue updating. Thus, an event for which the degree of confidence is higher is used as a hint to estimate θ .

Here, a simple example is considered. This case supposes that the physical quantity to be measured has a normal distribution, with mean $\mu_0 = 0$ and standard deviation $\sigma_0 = 1$. Consider estimating a physical quantity that has this distribution from experimentally measured values. In this case, the Bayes' theorem described above is used to update each measurement. For simplicity, it is assumed here that the standard deviation $\sigma_0 = 1$ is known and the mean is to be estimated. First, the two discrete events in Table 1 are the subject of estimation. The prior probabilities (confidence) are also shown. These two discrete values and the initial prior probabilities are assumed to be set based on the knowledge of people with experience. The sum of the prior probabilities needs to be 1.0. In this case, even though the population mean of 0.0 is correct, a confidence level as low as 0.3 is given to

Table 1 Two discrete events

Symbol	Events	Prior probability
C_1	$\mu = -0.1$	0.3
C_2	$\mu = 1.5$	0.7

-0.1, which is close to the correct answer, while a confidence level of 0.7 is set for 1.5, which is farther from the correct answer. Next is a simulation of how this confidence level is updated by trial experimentation. First, generate 10 $N(0.0, 1.0^2)$ normal random numbers and observe how the Bayesian update is performed with experimental data 10 times.

The i th random number is expressed as $rnd(i)$. The Bayesian update when the first experimental value $rnd(0)$ is obtained is performed using Eq. (7). The right side of this equation is all given quantities, as below.

$$\begin{aligned}
 P(C_1) &= 0.3 \\
 P(A|C_1) &= f(rnd(0) | -0.1, \sigma_0) \\
 P(C_2) &= 0.7 \\
 P(A|C_2) &= f(rnd(0) | 0.3, \sigma_0)
 \end{aligned} \tag{10}$$

Here, the first observed event A corresponds to obtaining experimental value $rnd(0)$. The probability density function $f(\cdot)$ is given by Eq. (1). This results in obtaining updated confidence value $P(C_1|A)$. Similarly, the C_2 updated confidence value $P(C_2|A)$ is obtained by Eq. (9). Furthermore, for the next experimental value $rnd(1)$, the next update can be performed as $P(C_1|A) \rightarrow P(C_1)$ and $P(C_2|A) \rightarrow P(C_2)$. The same procedure should be repeated after that. Figure 4 shows the change in posterior probability with Bayesian updating for the 10 experimental values. It shows that the C_1 confidence level, which is close to the true value, has risen, while the C_2 confidence level, which is far from the true value, has fallen. As a result, there is greater confidence that the true value is a C_1 event. Thus, Bayesian statistics is characterized as a method in

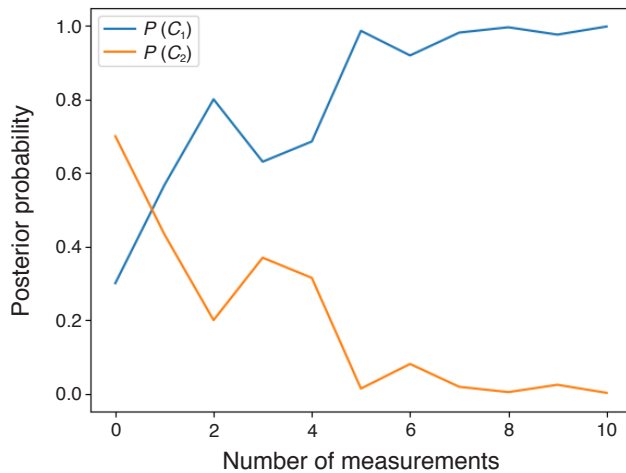


Fig. 4 Variation of posterior probability ($P(C_1) = 0.3$, $P(C_2) = 0.7$)

which predictions and confidence levels are set flexibly and subjectively when there is not a sufficient amount of sample data in the initial stage, after which the confidence level of the parameter is updated each time data is obtained based on a small number of experimental data points.

In the above example, a low confidence level of 0.3 was given for an initial prior distribution of -0.1, which was close to the correct answer, but if this confidence level had been set more appropriately high at 0.7, the results would be as shown in Fig. 5. Comparing this to Fig. 4 shows that the confidence level for the correct answer converges earlier to 1.0. Thus, if the initial prior distribution can be set more appropriately, it will converge to the correct answer more quickly, but if it is off the mark, the degree of convergence will not be as good. As this can result in less efficiency, the key is knowing how to appropriately set the initial prior distribution by referring to experienced people, past case studies, and other factors.

3.2.2. Estimation of continuous distribution when mean is unknown and variance is known

In the example above, instead of estimating the parameters of the burst probability density function to be estimated, a scheme was used in which the burst probability was assumed to be one of two discrete values and the degree of confidence in those discrete values was updated with Bayes' theorem. To extend this to a more general estimation method, the parameters themselves of the burst probability density function of Eq. (1) could be estimated. There are two parameters: the mean μ and the standard deviation σ . This section considers using Bayes' theorem to estimate these parameters. Although the theorem can be used for both parameters, for simplicity of explanation, only the mean μ is considered here. To make parameter estimation versatile enough for general use, it is advisable to use a format that expresses the level of confidence as a continuous distribution and then updates the continuous distribution, rather than use a format that estimates μ as

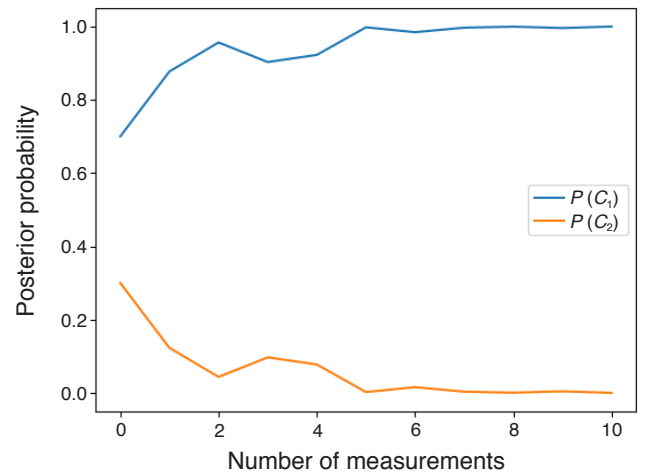


Fig. 5 Variation of posterior probability ($P(C_1) = 0.7$, $P(C_2) = 0.3$)

discrete values. That is to say, for a particular mean value μ , the burst probability density function is:

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] \quad (11)$$

This is called a likelihood function. The probability density function $g(\mu)$ is set as a continuous distribution expressing the uncertainty of μ . The approach of setting a continuous distribution will be discussed later, but first, it is set as a prior distribution, based on experience, as in the case of the discrete method. At this time, the posterior probability distribution when the observed value $X = (x_1, x_2, \dots, x_n)$ has been obtained can be expressed as follows based on the Bayesian update approach.

$$g'(\mu|X)d\mu = \frac{g(\mu) \prod_{i=1}^n f(x_i|\mu)d\mu}{\int_{-\infty}^{\infty} g(\mu) \prod_{i=1}^n f(x_i|\mu)d\mu} \quad (12)$$

Then, the posterior probability density function $g'(\mu|X)$ is derived based on Bayes' theorem. If the Bayesian update approach is applied each time an observed value is obtained, the continuous distribution function, which expresses the degree of confidence, can be expected to gradually converge to the true value. However, there is one great challenge with this. If one sets a completely arbitrary function as $g(\mu)$ for the prior distribution, the posterior distribution, $g'(\mu|X)$, would have no relation to the prior distribution. In this case, evaluating Eq. (12) becomes quite difficult. As computational capabilities have increased in recent years, it has become relatively easy to derive probability density functions numerically using the power of computers, even for complex expressions like that shown in Eq. (12). Markov Chain Monte Carlo Simulation (MCMC)⁽⁵⁾ is a typical method for doing this. In addition, software like Stan and PyMC3 are readily available to assist in such calculations. One approach is to use such software for arbitrary distributions, but fortunately, by attaching certain conditions to the prior probability, this process is extremely easy. That is the conjugate prior distribution approach. From the sake of general-purpose use and ease of processing, it is considered better to use the conjugate prior distribution approach, which is explained below.

The challenge now is that the prior distribution $g(\mu)$ and the posterior distribution $g'(\mu|X)$ have completely different distribution forms. It is known, however, that if one adopts a particular distribution form as $g(\mu)$ corresponding to the distribution form of $f(x|\mu)$ that is being addressed, the $g'(\mu|X)$ obtained by applying Bayes' theorem will have the same kind of distribution form. However, the parameters would be subject to change. When such a relationship is established, it is said that conjugacy is established, and a prior distribution that was set in this way is called a conjugate prior distribution. It is known that if $f(x|\mu)$ has a normal distribution, the conjugate prior distribution for μ has a normal distribution. In other words, $g'(\mu|X)$ also has a normal distribution.

The case here assumes that the random variable of

interest follows a normal distribution $N(\mu, \sigma^2)$, and that the variance σ^2 is known. A continuous distribution is set for the confidence level for parameter mean μ and a Bayesian update is performed. In this case, one uses the fact that the conjugate prior distribution for the mean is a normal distribution. In other words, if a normal distribution is set as the prior distribution of level of confidence for the mean, the posterior distribution will also be normal. When $N(\lambda_1, \tau_1^2)$ is set as the prior distribution, it is derived that the posterior distribution when a sample of $x = (x_1, x_2, \dots, x_n)$ is obtained is as follows⁽⁶⁾:

$$N(\lambda_2, \tau_2^2) \quad (13)$$

where

$$\hat{x} = \frac{\sum_{i=1}^n x_i}{n} \quad (14)$$

$$\lambda_2 = \frac{n\tau_1^2 \hat{x} + \sigma^2 \lambda_1}{n\tau_1^2 + \sigma^2}$$

$$\tau_2^2 = \frac{\sigma^2 \tau_1^2}{n\tau_1^2 + \sigma^2} \quad (15)$$

In other words, the two parameters of the posterior distribution that is a normal distribution can be easily evaluated from Eqs. (14) and (15). This posterior distribution can be used as the prior distribution the next time and thereafter.

The following is an example in which the correct answer for $f(x|\mu, \sigma)$ is $N(0, 1^2)$, which has a normal distribution. In this case, one starts the initial prior distribution for $N(\lambda, \tau^2)$ from $N(0.5, 1^2)$ and performs Bayesian updating. Assuming that 10 random number observations of $N(0, 1^2)$ are available at the time of measurement, one will see how the prior distribution changes as a result of Bayesian updating. The transition can be easily traced using Eqs. (14) and (15). Figure 6 shows the transition of the prior distribution for two Bayesian updates. It also shows that with each

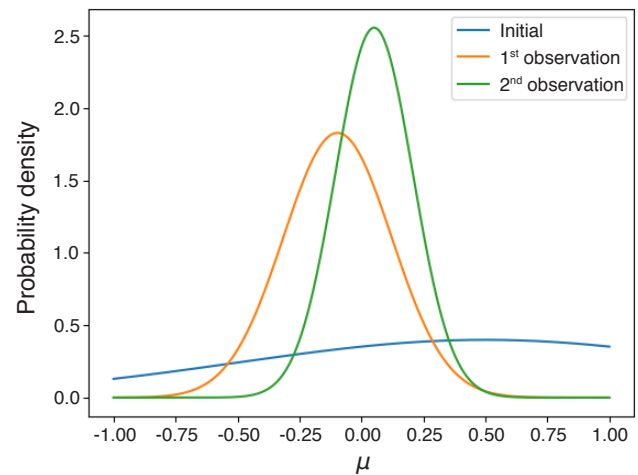


Fig. 6 Variation of prior distribution by Bayesian update

Bayesian update, the predicted prior distribution of the mean converges near the correct answer.

3.2.3. Estimation of continuous distribution when mean and variance are both unknown

This section considers a case in which both the mean μ and standard deviation σ are unknown and are estimated by Bayesian updating. In this case, the posterior probability distribution when the observed value $X = (x_1, x_2, \dots, x_n)$ has been obtained is evaluated with the following formula with Bayesian updating.

$$g'(\mu, \sigma|X) d\mu d\sigma = \frac{g(\mu, \sigma) \prod_{i=1}^n f(x_i|\mu, \sigma) d\mu d\sigma}{\int_0^\infty \int_{-\infty}^\infty g(\mu, \sigma) \prod_{i=1}^n f(x_i|\mu, \sigma) d\mu d\sigma} \quad (16)$$

Here, $g(\mu, \sigma)$ is a two-dimensional simultaneous probability density function that is a conjugate prior distribution and is found to be expressed as in Eqs. (18) and (19)^{(7),(8)}.

$$g(\mu, \sigma) = \pi(\mu|\sigma^2) \pi(\sigma^2) \quad (17)$$

$$\pi(\mu|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2/k_0}} \exp\left[-\frac{(\mu - \hat{\mu}_0)^2}{2\sigma^2/k_0}\right] \quad (18)$$

$$\pi(\sigma^2) = \frac{(\hat{\sigma}_0^2 v_0/2)^{v_0/2}}{\Gamma(v_0/2)} \frac{\exp\left(\frac{-v_0 \hat{\sigma}_0^2}{2\sigma^2}\right)}{(\sigma^2)^{1+v_0/2}} \quad (19)$$

Note that $\hat{\mu}_0$, k_0 , v_0 , and $\hat{\sigma}_0^2$ are parameters of the prior distribution, and in Eq. (19), the probability density function is an inverse gamma distribution expressed by the two parameters below.

$$f_{\text{Igamma}}(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} (1/x)^{\alpha+1} \exp(-\beta/x) \quad (20)$$

$$\alpha = v_0/2 \quad (21)$$

$$\beta = \hat{\sigma}_0^2 v_0/2 \quad (22)$$

It is derived that the parameters of the conjugate prior distribution are updated by the following calculation:

$$k' = k + n \quad (23)$$

$$m' = [km + n\hat{\mu}]/k' \quad (24)$$

$$v' = v + n \quad (25)$$

$$v(s')^2 = vs^2 + (n-1)S^2 + kn(\hat{\mu} - m)^2/k' \quad (26)$$

where $\hat{\mu}$ is the sample mean and S is the unbiased standard deviation of the sample data.

Here is an example that shows how the continuous distribution of confidence level is updated by Bayes' theorem in a case where the correct answer for the population is $N(7, 2^2)$ and 10 samples are obtained twice by Monte Carlo simulation. First, as the initial prior distribution of the confidence level, one begins far from the correct answers for the parameters, starting with $\hat{\mu}_0 = 4.0$, $k_0 = 1.0$, $v_0 = 2.0$, and $\hat{\sigma}_0^2 = 4.0$. Figure 7 shows the simultaneous probability density function for the initial prior distribution of the confidence level. The figure

shows that the coordinate values indicating the peak of this distribution are far from the correct answer, $(7, 2^2)$.

Figure 8 shows the result for the simultaneous probability density function obtained by generating 10 random numbers and performing Bayesian updating with Eq. (16). This could be interpreted as meaning that the coordinate values of the peak position are close to $(7, 2^2)$, but the figure shows that the density values are widely distributed around the peak value. Figure 9 shows the simultaneous probability density function obtained by Bayesian updating after generating the next 10 random numbers. The figure shows that the coordinate values of the peak asymptotically approach the correct solution, $(7, 2^2)$, and the density values are concentrated in that area. If Bayesian updating continues to be performed in the same manner each time samples are obtained, it is expected that the simultaneous probability density function values will be concentrated around the correct parameters.

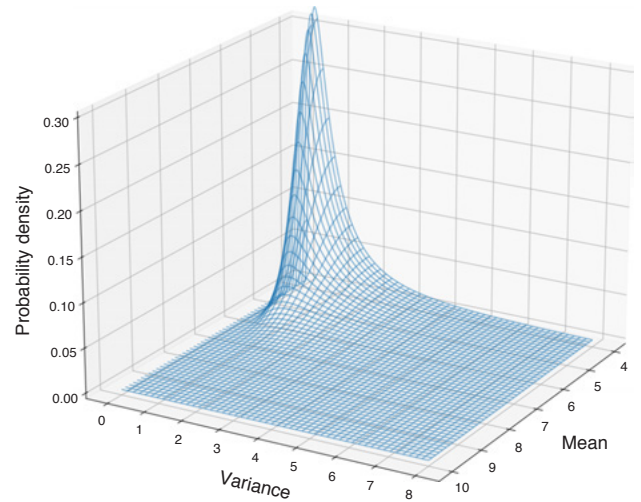


Fig. 7 Initial prior distribution

4. Application for Obtaining Statistical Data on Products with Design Changes

There are cases in which the product of interest is still in the development stage, but the statistical properties of the product are needed and statistical data needs to be obtained. If the product's manufacturing cost is high, it is already a challenge to increase the sample size. And if that product is in the development stage and its strength characteristics may change, there will likely be many challenges in obtaining the statistical properties of the product. Using Bayes' theorem is considered effective precisely in such cases. In other words, Bayes' theorem assumes that the initial prior distribution is as in Fig. 7. Bayesian updating can yield a posterior distribution like that in Fig. 8 in a single test, regardless of sample size. Then, assume that the design of the product will subsequently change. In general, when there are design changes during the development stage, it is considered

that in many cases the changes will not be substantial. In those cases, it is unlikely that the parameter values of the statistical properties will change dramatically. If so, the

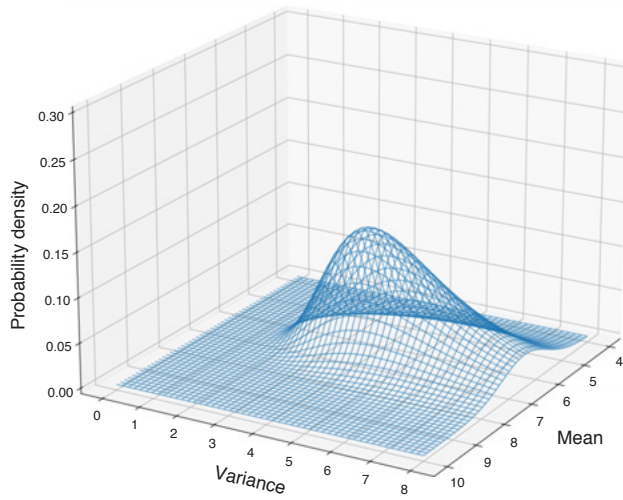


Fig. 8 Second prior distribution

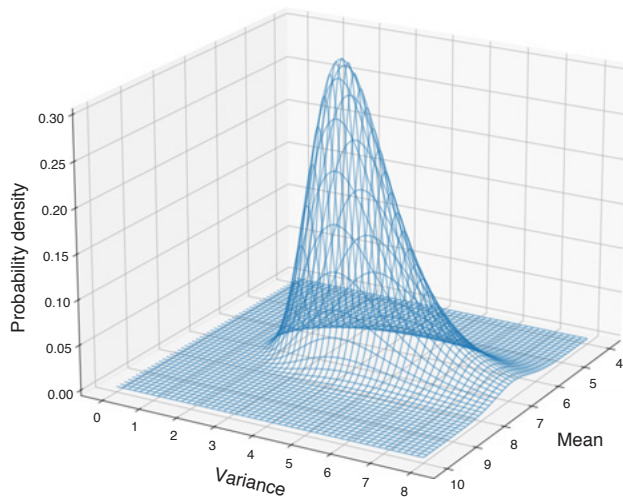


Fig. 9 Third prior distribution

distribution in Fig. 8 can be used as a prior distribution for estimating the statistical properties of the next product. Then, Bayesian updating can be expected to appropriately track changes in the statistical properties. However, if there is an extreme change in statistical properties, using the Fig. 8 distribution as a prior distribution may not be effective, in which case the distribution is reset, and one can proceed with Bayesian updating.

5. Example of Application for Pressure Equipment Maintenance

To give a concrete example of an industrial application of Bayes' theorem, this section presents its application to evaluate the probability of damage to pressure equipment. This information is necessary to perform risk-based inspection of such equipment. Risk-based inspection is one approach to maintaining piping, pressure vessels, and other pressure equipment. The main type of maintenance method in the past was to systematically conduct inspections at a fixed time, in a fixed location, and using fixed inspection methods. To eliminate the irrational aspects of this approach, risk-based inspection continuously reviews inspection plans using risk as an indicator. The first guideline to appear in this area was API581⁽⁹⁾ issued by the American Petroleum Institute. The corresponding Japanese standard is published by the High Pressure Institute of Japan⁽¹⁰⁾. This section presents an example of the application of Bayes' theorem to damage probability evaluation used in API581, while referring to reference⁽⁴⁾.

Figure 10 shows the general procedural flow for conducting a risk-based inspection. Since risk values are used as indicators when developing inspection programs, there are steps in which risk assessment is conducted, which are shown inside the dashed line of the figure. In this part, risk values are analyzed, and risk is defined as the consequence multiplied by likelihood (in bold frame) of damage. As Bayes' theorem is used to evaluate the part relating to likelihood of damage, this is introduced below.

Corrosion of metallic materials by internal fluids is one

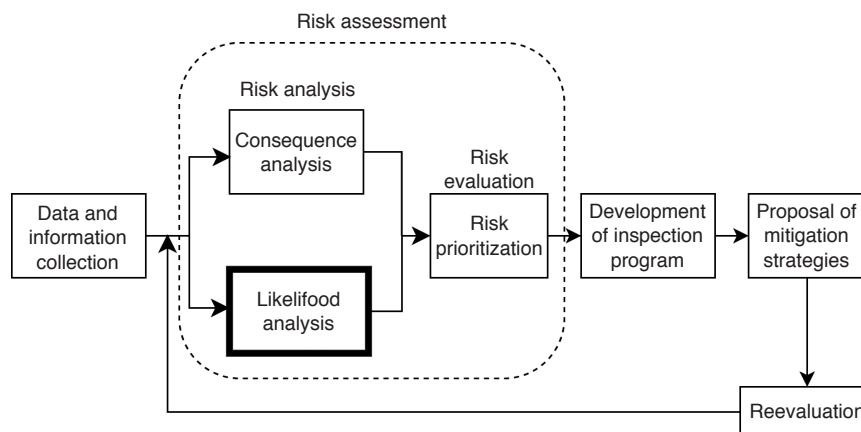


Fig. 10 Procedure of risk based inspection

Table 2 Definitions of damage states

Damage state	Definition
1	Damage does not exceed the predicted value of the damage model
2	Damage is at least equal to the predicted value of the damage rate model and at most twice that value
3	Damage is at least twice and at most four times the predicted value of the damage rate model

Table 4 Relation between inspection effectiveness and confidence

Damage rate state	Poorly effective	Fairly effective	Usually effective	Highly effective
1	0.33	0.5	0.7	0.90
2	0.33	0.3	0.2	0.09
3	0.33	0.2	0.1	0.01

Table 3 Allocation of initial confidence level

Actual damage rate range	Low reliability data	Moderate reliability data	High reliability data
Predicted rate or less	0.5	0.7	0.80
Predicted rate to two times rate	0.3	0.2	0.15
Two to four times predicted rate	0.2	0.1	0.05

of the main damage modes in the maintenance of piping and pressure vessels. As long as equipment is operated under the same operating conditions and environment, the rate of corrosion is expected not to change greatly because corrosion progresses steadily. Damage occurs when corrosion penetrates through the thickness of the plate, allowing internal fluid to leak. Therefore, one can evaluate the likelihood of damage; that is, the damage probability, based on the percentage of plate thickness that remains. If the residual wall thickness is measured by appropriate non-destructive testing methods and one assumes that corrosion progresses steadily, one can also predict future wall thinning. Based on this, one can perform a damage probability evaluation and review inspection plans while referring to the results of the risk assessment. In this case, it is important to accurately evaluate the corrosion rate by appropriate non-destructive testing methods. However, if measurements are made with less accurate non-destructive testing methods, the accuracy of the evaluated corrosion rate will decline correspondingly. In that sense, it is important to evaluate the effectiveness of non-destructive testing methods, which is determined by a combination of the type of damage targeted and the non-destructive testing methods. This is called inspection effectiveness, and the accuracy of the evaluated corrosion rate requires an evaluation of inspection effectiveness. However, inspection effectiveness cannot be expressed as an objective quantity, and inevitably, the evaluation will contain some subjectivity. By applying Bayes' theorem to the evaluation of corrosion rates, one can incorporate an evaluation of an ambiguous quantity like inspection effectiveness and then evaluate the accuracy of corrosion rates. A concrete explanation follows.

Step 1

It is necessary to draw some conclusions from past findings regarding the target damage mechanism. Typical examples include whole-surface corrosion and localized corrosion, but here it is decided to use whole-surface corrosion as an example. Although it is generally a

challenge to obtain corrosion rate data in the early stages, if historical inspection data is available, it may be used as a rough estimate. If historical data does not exist, use the predicted wall-thinning rate based on the literature. Here, 0.2 mm/year is used as a possible prediction based on values found in the literature.

Step 2

Determine whether the level of corrosion rate reliability set in step 1 corresponds to low, moderate, or high. If there is complete field data, the confidence level can be classified as high, but if there is no existing data and it is inferred, as in this example, it is classified as low. The API guideline provides detailed classification criteria. What Bayes' theorem seeks to predict is not corrosion loss itself, but the accuracy of the thinning rate currently in use, which it classifies into three states. Perform Bayesian updating of the degree of confidence to determine which of the three states applies. The three accuracy states are shown in Table 2. Table 3 shows the initial degrees of confidence, or prior probabilities, for the three damage states used when applying Bayes' theorem.

Step 3

Perform Bayesian updating of the degree of confidence in the damage conditions defined in Table 2. The damage state indicates the degree of ambiguity of the evaluation result as found by inspection, with larger values indicating a greater degree of ambiguity. The degree of confidence depends on the inspection effectiveness described above. Damage evaluation accuracy depends on the non-destructive testing method used and the control volume ratio. A 100% volume inspection with a highly accurate non-destructive testing method would increase the degree of confidence in damage state 1 in Table 2. This measure of effectiveness is the inspection effectiveness, and the degree of confidence varies with the inspection effectiveness. For each level of inspection effectiveness, the relationship between the damage state and the degree of confidence that it can be accurately measured are given in Table 4. This

table appears to have been developed based on the opinions of experts and the like.

These values signify the conditional probability of each damage state for the subsequent Bayesian update. In other words, the table shows the probability that the predicted rate will be observed, on the condition that the current damage state will be one of the three states.

Step 4

Perform Bayesian updating to determine how the degree of confidence in each damage state changes as a result of performing inspections. That is, update the degree of confidence by using Bayes' theorem based on the results at the time of inspection. Apply Bayes' theorem according to the following equation:

$$P[A_i|B_k] = \frac{P[B_k|A_i]P[A_i]}{\sum_j P[B_k|A_j]P[A_j]} \quad (27)$$

where the symbols have the following meanings:

- $P[A_i]$ Prior probability: Degree of confidence that the damage state is i
- $P[A_i|B_k]$ Posterior probability: The updated degree of confidence when the event at the time of inspection was A_k (denoted as B_k in the equation)
- $P[B_k|A_i]$ Conditional probability: The conditional probability that damage state k will be observed when the current damage state is i . In the API, this is mapped to the inspection effectiveness. Table 4 shows values for $i = 1, 2, 3; k = 1$, but there are separate tables for $i = 1, 2, 3; k = 2$ and $i = 1, 2, 3; k = 3$ in the standard.

Assuming that during the initial inspection the predicted corrosion rate was observed using *Usually effective* non-destructive testing, the Bayesian update procedure would be as follows.

1. Since the prior probabilities are based on data from the literature, use the low reliability data from Table 3 as $P[A_i]$.
2. Use the values from the relevant part of Table 4 based on the fact that the *Usually effective* predicted corrosion rate was confirmed as the inspection effectiveness.
3. Perform evaluation assuming that the results at the time of inspection showed the damage rate to be as predicted, that is, B_1 .

Therefore, the correspondence of the variables in Eq. (27) is as follows.

$$P[A_1] = 0.5, P[A_2] = 0.3, P[A_3] = 0.2 \\ P[B_1|A_1] = 0.7, P[B_1|A_2] = 0.2, P[B_1|A_3] = 0.1 \quad (28)$$

Substituting these values into Eq. (27) yields the

Table 5 Confidence level after first inspection

Damage rate state	After a usually effective inspection—
1	0.814
2	0.140
3	0.046

following:

$$P[A_1|B_1] = \frac{0.7 \times 0.5}{0.7 \times 0.5 + 0.2 \times 0.3 + 0.1 \times 0.2} = 0.814 \\ P[A_2|B_1] = \frac{0.2 \times 0.3}{0.7 \times 0.5 + 0.2 \times 0.3 + 0.1 \times 0.2} = 0.140 \\ P[A_3|B_1] = \frac{0.1 \times 0.2}{0.7 \times 0.5 + 0.2 \times 0.3 + 0.1 \times 0.2} = 0.046 \quad (29)$$

That is, Table 5 is obtained.

The result is that the degree of confidence that the damage state is 1 has increased. Thus, Bayes' theorem updates the degree of confidence in the current damage rate state by incorporating even a subjective measure of inspection effectiveness. If this procedure is followed at each subsequent inspection, it will be possible to determine the current damage rate state.

Step 5

Based on the predicted rate of 0.2 mm/year for damage state 1, a worst-case predicted rate of 0.4 mm/year for state 2, and a worst-case predicted rate of 0.8 mm/year for state 3, the probability of damage at the next inspection can be evaluated for each damage state. The procedure is omitted here because it is detailed, but the overall probability of damage is evaluated based on the probability of damage in each state and its degree of confidence in Table 5. The risk is then assessed by multiplying this and an evaluation of the consequence.

6. Conclusion

This paper has shown that using Bayes' theorem is effective when the objective is to obtain statistical properties of bursting strength for equipment like composite hydrogen tanks. The main reason for this is that it is so challenging to obtain large amounts of bursting strength data for reasons of cost and labor, and thus it is a challenge to evaluate bursting strength by sample statistics. With Bayes' theorem, it is possible to use small sample data to increase the degree of confidence in the parameter by Bayesian updating. However, setting the initial prior distribution, which is to say the initial confidence level, involves arbitrariness. Depending on the circumstances, therefore, there could be aspects that are hard to handle, such as setting the initial prior distribution subjectively. This paper used examples to explain Bayesian statistics, from the basics to application. The author hopes this paper will make at least a small contribution to obtaining small-sample statistical data.

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